

# A characterization of $L_2$ mixing and hypercontractivity via hitting times and maximal inequalities

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## Abstract

There are several works characterizing the total-variation mixing time of a reversible Markov chain in term of natural probabilistic concepts such as stopping times and hitting times. In contrast, there is no known analog for the  $L_2$  mixing time,  $\tau_2$  (while there are sophisticated analytic tools to bound  $\tau_2$ , in general they do not determine  $\tau_2$  up to a constant factor and they lack a probabilistic interpretation). In this work we show that  $\tau_2$  can be characterized up to a constant factor using hitting times distributions. We also derive a new extremal characterization of the Log-Sobolev constant,  $c_{LS}$ , as a weighted version of the spectral gap. This characterization yields a probabilistic interpretation of  $c_{LS}$  in terms of a hitting time version of hypercontractivity. As applications of our results, we show that (1) for every reversible Markov chain,  $\tau_2$  is robust under addition of self-loops with bounded weights, and (2) for weighted nearest neighbor random walks on trees,  $\tau_2$  is robust under bounded perturbations of the edge weights.

**Keywords:** Mixing-time, finite reversible Markov chains, maximal inequalities, hitting times, hypercontractivity, Log-Sobolev inequalities, relative entropy, robustness of mixing times.

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# 1 Introduction

There are numerous essentially equivalent characterizations of mixing in  $L_1$  (e.g. [1, Theorem 4.6] and [15]) of a finite reversible Markov chain. Some involve natural probabilistic concepts such as couplings, stopping times and hitting times (see § 1.8). In contrast, (paraphrasing Aldous and Fill [1] last sentence of page 155, which mentions that there is no  $L_2$  counterpart to [1, Theorem 4.6]) while there are several sophisticated analytic and geometric tools for bounding the  $L_2$  mixing time,  $\tau_2$ , none of them has a probabilistic interpretation, and none of them determines  $\tau_2$  up to a constant factor.

In this work we provide probabilistic characterizations in terms of hitting times distributions for the  $L_2$  mixing time and also for the mixing time in relative entropy,  $\tau_{\text{Ent}}$  (see (1.13) and (1.17) for definitions), of a reversible Markov chain (Theorem 1.1).

While the spectral gap is a natural and simple parameter, the *Log-Sobolev constant* (see (1.15)),  $c_{\text{LS}}$ , is a more involved quantity. When one first encounters  $c_{\text{LS}}$ , it may seem like an artificial parameter that “magically” gives good bounds on  $\tau_2$ . We give a new extremal characterization of the Log-Sobolev constant as a weighted version of the spectral gap. This characterization gives a direct link between  $c_{\text{LS}}$  and  $\tau_2$  (answering a question asked by James Lee, see Remark 1.2) and can be interpreted probabilistically as a hitting-time version of hypercontractivity (see the discussion in § 5.2 and Conjecture 5.2).

## 1.1 Characterizations of $\tau_2$ and $\tau_{\text{Ent}}$ using hitting times

We now describe the aforementioned characterizations of  $\tau_2$  and  $\tau_{\text{Ent}}$ . More refined versions will be given later on in Theorems 4.1 and 6.1. Recall that for a Markov chain  $(X_t)_{t \geq 0}$  with state space  $\Omega$ , the *hitting-time* of a set  $A \subset \Omega$  is  $T_A := \inf\{t : X_t \in A\}$ . We say that  $A$  is connected if  $\mathbb{P}_a[T_b < T_{A^c}] > 0$ , for all  $a, b \in A$ . We denote by  $\text{Con}_\delta$  the collection of all connected sets  $A$  satisfying  $\pi(A) \leq \delta$ , where throughout,  $\pi$  shall denote the stationary distribution of the chain. Denote

$$\rho := \max_{x \in \Omega} \rho_x \quad \text{and} \quad \rho_{\text{Ent}} := \max_{x \in \Omega} \rho_{\text{Ent},x}, \quad \text{where} \quad (1.1)$$

$$\begin{aligned} \rho_x &:= \min\{t : \mathbb{P}_x[T_{A^c} > t] \leq \pi(A) + \frac{1}{2} \sqrt{\pi(A)\pi(A^c)} \text{ for all } A \in \text{Con}_{1/2}\}, \\ \rho_{\text{Ent},x} &:= \min\{t : \mathbb{P}_x[T_{A^c} > t] \leq \frac{C}{|\log \pi(A)|}, \text{ for all } A \in \text{Con}_{1/2}\}, \end{aligned} \quad (1.2)$$

for some absolute constant  $C > 0$  to be determined later. Note that allowing  $A$  above to range over all  $A \subset \Omega$  such that  $\pi(A) \leq 1/2$  does not change the values of  $\rho_x$  and  $\rho_{\text{Ent},x}$ .

**Theorem 1.1.** *There exist absolute constants  $C_1, C_2, C_3$  such that for every irreducible reversible Markov chain on a finite state space*

$$\rho \leq \tau_2 \leq \rho + C_1/c_{\text{LS}} \leq C_2\rho. \quad (1.3)$$

$$\rho_{\text{Ent}} \leq \tau_{\text{Ent}} \leq C_3\rho_{\text{Ent}}. \quad (1.4)$$

Note that in the definitions of  $\rho$  and  $\rho_{\text{Ent}}$ , the smaller  $A$  is, the smaller we require the chance of not escaping it by time  $\rho$  or  $\rho_{\text{Ent}}$ , respectively, to be. In other words, the smaller  $A$  is, the higher the “penalty” we assign to the case the chain did not escape from it. As we explain in § 1.7, the first inequalities in (1.3)-(1.4) are easy and even somewhat “naive”.

## 1.2 A new extremal characterization of the Log-Sobolev time.

A lot of attention has been focused on inequalities that interpolate between the Log-Sobolev inequality and the Poincaré (spectral gap) inequality (e.g. [3, 13]). Using similar ideas as described above we prove a new extremal characterization (up to a constant factor) of the *Log-Sobolev constant* (Theorems 1.2),  $c_{\text{LS}}$  (see (1.15) for a definition). The *Log-Sobolev time* is defined as  $t_{\text{LS}} := 1/c_{\text{LS}}$ .

The aforementioned characterization has a relatively simple form which does not involve any entropy. Instead, it describes the Log-Sobolev constant as a weighted version of the spectral gap. This characterization provides some insights regarding the hierarchy of the aforementioned inequalities. Before presenting it, we first need a few definitions.

The *time-reversal* of  $P$  is defined as  $P^*(x, y) := \pi(y)P(y, x)/\pi(x)$ . This is the dual operator of  $P$  w.r.t.  $L_2(\Omega, \pi)$ . We say  $P$  is *reversible* if  $P = P^*$ . Denote  $Q := (P + P^*)/2$ . Note that  $Q = Q^*$ . The *spectral gap* of  $P$ ,  $\lambda$ , is defined as the smallest non-zero eigenvalue of  $I - Q$ . We define  $t_{\text{rel}} := 1/\lambda$ . Let  $A \subsetneq \Omega$ . Let  $Q_A$  (resp.  $P_A$ ) be the restriction of  $Q$  (resp.  $P$ ) to  $A$ . Note that  $Q_A$  and  $P_A$  are substochastic. The spectral gap of  $P_A$ ,  $\lambda(A)$ , is defined as the minimal eigenvalue of  $I - Q_A$ . Denote  $t_{\text{rel}}(A) := 1/\lambda(A)$ . Denote

$$\kappa := 1/\alpha, \quad \alpha := \min_{A \in \text{Con}_{1/2}} \alpha(A), \quad \text{where} \quad \alpha(A) := \lambda(A)/|\log \pi(A)|. \quad (1.5)$$

As mentioned earlier,  $\alpha$  is a weighted version of  $\lambda$  since ([1, Lemma 4.39] and [8, (1.4)])

$$\lambda/2 \leq \min_{A \in \text{Con}_{1/2}} \lambda(A) \leq \lambda, \quad \text{and so} \quad t_{\text{rel}} \log 2 \leq \kappa. \quad (1.6)$$

**Theorem 1.2.** *For every irreducible Markov chain on a finite state space*

$$\kappa \leq t_{\text{LS}} \leq 2(\kappa + t_{\text{rel}}(1 + \log 49)) \leq 2(1 + (1 + \log 49)/\log 2)\kappa < 17\kappa. \quad (1.7)$$

**Remark 1.1.** *The inequality  $\kappa \leq t_{\text{LS}}$  is easy. See Lemma 4.2 in [8] for a stronger inequality. The harder and more interesting direction is  $t_{\text{LS}} \leq C\kappa$ , which is an improvement over the well-known inequality  $t_{\text{LS}} \leq t_{\text{rel}} \frac{\log[1/\pi_* - 1]}{1 - 2\pi_*}$ , where  $\pi_* := \min_{x \in \Omega} \pi(x)$  [5, Corollary A.4].*

**Remark 1.2.** *Despite the fact that  $t_{\text{LS}}$  is a geometric quantity, Logarithmic Sobolev inequalities have a strong analytic flavor and little probabilistic interpretation. For instance, the proof of the inequality  $t_{\text{LS}} \leq 2\tau_2(1/e)$  [5, Corollary 3.11] (where  $\tau_2(\varepsilon)$  is the  $L_2$  mixing time defined in (1.13)) relies on Stein’s interpolation Theorem for a family of analytic operators. Our analysis yields a probabilistic proof of the fact that  $t_{\text{LS}} \leq C\tau_\infty$  for reversible chains (The problem of finding such a proof was posed by James Lee at the Simons institute in 2015.) Indeed by Theorem 1.2 and (3.2),  $t_{\text{LS}}/17 \leq \kappa \leq 3\rho \leq 3\tau_2$  (this second inequality is relatively easy, and is obtained by analyzing hitting times, rather than by analytic tools). As we show in § 1.7, the inequality  $\rho \leq \tau_2$  also has a probabilistic interpretation.*

### 1.3 Robustness of $\tau_2$ under addition of self-loops of bounded weights.

**Corollary 1.3.** *Let  $(X_t)$  be a reversible irreducible continuous-time Markov chain on a finite state space  $\Omega$  with generator  $G$ . Let  $(\tilde{X}_t)$  be a chain with generator  $\tilde{G}$  obtained by multiplying for all  $x \in \Omega$  the  $x$ th row of  $G$  by some  $r_x \in (1/M, M)$  (for some  $M \geq 1$ ). Then for some absolute constant  $C$  the corresponding  $L_2$  mixing times satisfy*

$$\tilde{\tau}_2 / (CM \log M) \leq \tau_2 \leq (CM \log M) \tilde{\tau}_2. \quad (1.8)$$

This corollary, proved in § 7.1 is an analog of [15, Corollary 9.5], which gives the corresponding statement for  $\tau_1$ . While the statement is extremely intuitive, surprisingly, it was recently shown that it may fail for simple random walk on an Eulerian digraph [4, Theorem 1.5].

Observe that the generator  $G$  of a reversible chain on a finite state space  $\Omega$ , can be written as  $r(P - I)$ , where  $P$  is the transition matrix of some nearest neighbor weighted random walk on a network which may contain some weighted self-loops. The operation of multiplying the  $x$ th row of  $G$  by some  $r_x \in (1/M, M)$  for all  $x \in \Omega$  is the same as changing  $r$  above by some constant factor and changing the weights of the self-loops by a constant factor.

**Remark 1.4.** *Similarly, one can show that under reversibility the  $L_2$  mixing time in the discrete-time lazy setup is robust under changes of the holding probabilities. More precisely, for every  $\delta \in (0, 1/2]$  if we consider a chain that for all  $x \in \Omega$ , when at state  $x$  it stays put w.p.  $\delta \leq a(x) \leq 1 - \delta$  and otherwise moves to state  $y$  w.p.  $P(x, y)$  (where  $P$  is reversible), then its  $L_2$  mixing time can only differ from the  $L_2$  mixing time of the chain with  $a(x) = 1/2$  for all  $x$ , by a factor of  $C\delta^{-1}|\log \delta|$ .*

### 1.4 Robustness of $\tau_\infty$ for trees.

Recall that for reversible chains the  $L_2$  mixing time,  $\tau_2$ , determines the  $L_p$ -mixing time up to a factor  $c_p$  for all  $1 < p \leq \infty$  (see (1.14)). Denote the  $L_p$  mixing time of simple random walk on a finite connected simple graph  $G$  by  $\tau_p(G)$ . Kozma [10] made the following conjecture:

**Conjecture 1.5** ([10]). *Let  $G$  and  $H$  be two finite  $K$ -roughly isometric graphs of maximal degree  $\leq d$ . Then*

$$\tau_\infty(G) \leq C(K, d)\tau_\infty(H). \quad (1.9)$$

It is well-known that (1.9) is true if one replaces  $\tau_\infty$  with  $t_{\text{LS}}$  (e.g. [5, Lemma 3.4]). Ding and Peres [6] showed that (1.9) is false if one replaces  $\tau_\infty$  with  $\tau_1$ . In part, their analysis relied on the fact that the total variation mixing time can be related to hitting times, which may be sensitive to small changes in the geometry. Hence it is natural to expect that a description of  $\tau_\infty$  in terms of hitting times might shed some light on Conjecture 1.5. Indeed this was one of the main motivations for this work. In [9] the first author constructed a counterexample to Conjecture 1.5, where also there the key is sensitivity of hitting times.

Peres and Sousi [15, Theorem 9.1] showed that for weighted nearest neighbor random walks on trees (see § 7.2 for a definition),  $\tau_1$  can change only by a constant factor, as a result of a bounded perturbation of the edge weights. As an application of Theorem 1.1 we extend their result to the case of  $\tau_2$ .

**Theorem 1.3.** *There exists an absolute constant  $C$  such that for every finite tree  $\mathcal{T} = (V, E)$  with some edge weights  $(w_e)_{e \in E}$ , the corresponding random walk satisfies that*

$$\max(\tau_1, t_{\text{LS}}/4) \leq \tau_2 \leq \tau_1 + C \max(t_{\text{LS}}, \sqrt{t_{\text{LS}}\tau_1}), \quad (1.10)$$

*Consequently, if  $(w'_e)_{e \in E}, (w_e)_{e \in E}$  are two edge weights such that  $1/M \leq w_e/w'_e \leq M$  for all  $e \in E$ , then there exists a constant  $C_M$  (depending only on  $M$ ) such that the corresponding  $L_\infty$  mixing times,  $\tau_\infty$  and  $\tau'_\infty$ , satisfy*

$$\tau'_\infty/C_M \leq \tau_\infty \leq C_M \tau'_\infty. \quad (1.11)$$

**Remark 1.6.** *Since  $t_{\text{LS}}$  is robust under a bounded perturbation of the edge weights (e.g. [5, Lemma 3.3]), indeed (1.11) follows from (1.10) in conjunction with the aforementioned  $L_1$  robustness of trees (and the fact that  $\tau_2 \leq \tau_\infty \leq 2\tau_2$ , see (1.14)).*

## 1.5 Basic definitions and notation.

Generically, we shall denote the state space of a Markov chain  $(X_t)$  by  $\Omega$  and its stationary distribution by  $\pi$ . We denote such a chain by  $(\Omega, P, \pi)$ . We say that the chain is finite, whenever  $\Omega$  is finite. The continuous-time version of a chain is a continuous-time Markov chain whose distribution at time  $t$  is given by the heat kernel  $H_t := e^{-t(I-P)}$ . We denote  $h_t(x, y) := H_t(x, y)/\pi(y)$ . All of our results can be extended to the setup of discrete-time chains. The only difference is that one has to replace  $\lambda$  and  $t_{\text{rel}}$  by  $\bar{\lambda} := \min(\lambda, 2 - \lambda_{|\Omega|})$  and  $1/\bar{\lambda}$ , resp., where  $\lambda_{|\Omega|}$  is the maximal eigenvalue of  $I - P$ . To avoid repetitions we present our results only in the continuous-time setup. We note that most of our results can be extended to the general setup of ergodic Markov chains. However, working in such generality leads to many technical difficulties which we chose to avoid for the sake of clarity of presentation.

We denote by  $P_x^t$  (resp.  $P_x$ ) the distribution of  $X_t$  (resp.  $(X_t)_{t \geq 0}$ ), given that the initial state is  $x$ . The  $L_p$  norm of a function  $f \in \mathbb{R}^\Omega$  is  $\|f\|_p := (\mathbb{E}_\pi[|f|^p])^{1/p}$  for  $1 \leq p < \infty$  (where  $\mathbb{E}_\pi[h] := \sum_x \pi(x)h(x)$ ) and  $\|f\|_\infty := \max_x |f(x)|$ . The  $L_p$  norm of a signed measure  $\sigma$  is

$$\|\sigma\|_{p,\pi} := \|\sigma/\pi\|_p, \quad \text{where} \quad (\sigma/\pi)(x) = \sigma(x)/\pi(x).$$

We denote the worst case  $L_p$  distance at time  $t$  by  $d_p(t) := \max_x d_{p,x}(t)$ , where  $d_{p,x}(t) := \|P_x^t - \pi\|_{p,\pi}$ . Under reversibility for all  $x \in \Omega$  and  $t \geq 0$  (e.g. (2.2) in [8])

$$d_{2,x}^2(t) = h_{2t}(x, x) - 1, \quad d_\infty(t) = \max_y h_t(y, y) - 1. \quad (1.12)$$

The  $\varepsilon$ - $L_p$ -**mixing-time** of the chain (resp. for a fixed starting state  $x$ ) is defined as

$$\tau_p(\varepsilon) := \max_x \tau_{p,x}(\varepsilon), \quad \text{where} \quad \tau_{p,x}(\varepsilon) := \min\{t : d_{p,x}(t) \leq \varepsilon\}. \quad (1.13)$$

When  $\varepsilon = 1/2$  we omit it from the above notation. Let  $m_p := 1 + \lceil (2-p)/(2(p-1)) \rceil$ . It follows from (1.12), Jensen's inequality and the Reisz-Thorin interpolation Theorem that for reversible chains, the  $L_p$  mixing times can be compared as follows (e.g. [16, Lemma 2.4.6]):

$$\begin{aligned} \tau_2(a) \leq \tau_p(a) \leq 2\tau_2(\sqrt{a}) = \tau_\infty(a) & \quad \text{for all } p \in (2, \infty] \text{ and } a > 0, \\ \frac{1}{m_p} \tau_2(a^{m_p}) \leq \tau_p(a) \leq \tau_2(a) & \quad \text{for all } p \in (1, 2) \text{ and } a > 0, \end{aligned} \quad (1.14)$$

Hence for all  $1 < p \leq \infty$  the  $L_p$  convergence profile is determined by that of  $L_2$ .

With the convention  $0 \log 0 = 0$ , for all non-zero  $f, g \in \mathbb{R}_+^\Omega$  we define  $\langle f, g \rangle_\pi := \mathbb{E}_\pi[fg]$ ,

$$\text{Ent}_\pi(f) := \mathbb{E}_\pi[f \log f] - \mathbb{E}_\pi[f] \log \mathbb{E}_\pi[f] = \mathbb{E}_\pi[f \log(f/\mathbb{E}_\pi[f])],$$

$\mathcal{E}(f, g) := \langle (I - Q)f, g \rangle_\pi$  and  $\mathcal{E}(f) := \mathcal{E}(f, f)$ . The *Log-Sobolev constant* of the chain is

$$c_{\text{LS}} := \inf\{\mathcal{E}(f)/\text{Ent}_\pi(f^2) : f \text{ is non-constant}\}. \quad (1.15)$$

Recall that  $t_{\text{LS}} := 1/c_{\text{LS}}$ . It is always the case that  $t_{\text{LS}} \geq 2t_{\text{rel}}$  (e.g. [5, Lemma 3.1]).

The relative entropy of a distribution  $\mu$  w.r.t.  $\pi$  is defined as

$$D(\mu||\pi) := \sum_x \mu(x) \log(\mu(x)/\pi(x)) = \text{Ent}_\pi(\mu/\pi), \quad (1.16)$$

The mixing time in relative entropy is defined as

$$\tau_{\text{Ent},x} := \inf\{t : D(P_x^t||\pi) \leq 1/2\} \quad \text{and} \quad \tau_{\text{Ent}} = \max_x \tau_{\text{Ent},x}. \quad (1.17)$$

Finally, we note that while some effort was made to make most constants explicit in order to demonstrate that they are not large, we did not attempt to optimize constants. We use the convention that  $C, C', C_1, \dots$  (resp.  $c, c', c_1, \dots$ ) denote positive absolute constants which are sufficiently large (resp. small). Different appearances of the same constant at different places may refer to different numeric values.

## 1.6 Organization of this work

In § 1.6 we prove the lower bounds on  $\tau_2$  and  $\tau_{\text{Ent}}$  from (1.3) and (1.4) and present a sketch of the proof of the upper bound on  $\tau_2$  from (1.3). In § 2 and 3 we present some auxiliary results about maximal inequalities and hitting times. In § 4 and § 6 we prove slightly more refined versions of (1.3) and (1.4), resp.. In § 5 we prove Theorem 1.2. In § 7 we prove Theorem 1.3. We conclude with some open problems in § 8.

## 1.7 An overview of our approach

We start with an illustrating example: if  $\mathbb{P}_x[T_{A^c} > t] > 3\pi(A)/2$  for some set  $A$ , then

$$[H_t(x, A) - \pi(A)]/\pi(A) \geq [\mathbb{P}_x[T_{A^c} > t] - \pi(A)]/\pi(A) > 1/2.$$

Denote  $\pi$  conditioned on  $A$  by  $\pi_A(a) := 1_{a \in A}\pi(a)/\pi(A)$ . Finally, note that

$$d_{\infty,x}(t) \geq \max_{a \in A} h_t(x, a) - 1 \geq \sum \pi_A(a)(h_t(x, a) - 1) = [H_t(x, A) - \pi(A)]/\pi(A) > 1/2.$$

Hence  $\tau_{\infty,x} \geq \min\{t : \mathbb{P}_x[T_{A^c} > t] \leq 3\pi(A)/2, \text{ for all } A\}$ .

This generalizes as follows. Let  $A \subsetneq \Omega$ ,  $x \in \Omega$ ,  $t > 0$  and  $\delta \in (0, 1)$ . Let  $\mathcal{P}_{A,\delta}$  be the collection of all distributions  $\mu$  on  $\Omega$ , satisfying that  $\mu(A) \geq \pi(A) + \delta\pi(A^c)$ . Clearly, if  $\mathbb{P}_x[T_{A^c} > t] \geq \pi(A) + \delta\pi(A^c)$ , then  $\mathbb{P}_x^t \in \mathcal{P}_{A,\delta}$ . Note that

$$\nu_{A,\delta} := \delta\pi_A + (1 - \delta)\pi \in \mathcal{P}_{A,\delta}.$$

Moreover,  $\delta := \min\{\delta' : \nu_{A,\delta'} \in \mathcal{P}_{A,\delta}\}$ . It is thus intuitive that for a convex distance function between distributions,  $\nu_{A,\delta}$  is the closest distribution to  $\pi$  in  $\mathcal{P}_{A,\delta}$ .

**Proposition 1.7.** *Let  $(\Omega, P, \pi)$  be some finite irreducible Markov chain. Let  $A \subsetneq \Omega$ . Denote  $\nu_{A,\delta} := \delta\pi_A + (1-\delta)\pi$ . Then for all  $\delta \in (0, 1)$ ,*

$$\begin{aligned} \min_{\mu \in \mathcal{P}_{A,\delta}} \|\mu - \pi\|_{2,\pi} &= \|\nu_{A,\delta} - \pi\|_{2,\pi} = \delta \sqrt{\pi(A^c)/\pi(A)}. \\ \min_{\mu \in \mathcal{P}_{A,\delta}} D(\mu||\pi) &= D(\nu_{A,\delta}||\pi) = u(\pi(A), \delta), \end{aligned} \quad (1.18)$$

where  $u(x, y) := [y + x(1-y)] \log(1 + \frac{y(1-x)}{x}) + (1-y)(1-x) \log(1-y)$ .

*Proof.* The first equality in both lines can be verified using Lagrange multipliers. The second equality in both lines is straightforward.  $\square$

Proposition 1.7 motivates the definitions in (1.2). We argue that (1.18), implies both (1.3)-(1.4) by making suitable substitutes for  $\delta$  in (1.18). For (1.3) substitute  $\delta = \frac{1}{2} \sqrt{\frac{\pi(A)}{\pi(A^c)}}$  in the first line of (1.18). For every  $x \in \Omega$  and  $t < \rho_x$  there is some  $A \in \text{Con}_{1/2}$  such that

$$\mathbb{P}_x[T_{A^c} > t] > \pi(A) + \frac{1}{2} \sqrt{\pi(A)\pi(A^c)} = \pi(A) + \delta\pi(A^c),$$

where the equality follows by our choice of  $\delta$ . As mentioned above, this implies that  $\mathbb{P}_x^t \in \mathcal{P}_{A,\delta'}$  for some  $\delta' > \delta$  and so by (1.18) and the choice of  $\delta$ ,  $\|\mathbb{P}_x^t - \pi\|_{2,\pi} > 1/2$ . For (1.4), it is not hard to verify that for some  $C', C > 0$ , we have that  $u(x, \min(\frac{C'}{|\log x|}, 1)) \geq 1/2$  and  $x + \frac{C'}{|\log x|}(1-x) \leq \frac{C}{|\log x|}$  for all  $x \leq 1/2$ . Substituting  $\delta = \frac{C'}{|\log \pi(A)|}$  in the second line of (1.18) implies (1.4) in a similar manner to the above derivation of (1.3).

We now explain the idea behind the proof of the upper bound on  $\tau_2$  from (1.3). Let  $x \in \Omega$ . Denote  $t := \rho_x + 8\kappa + 6t_{\text{rel}} \log 2$ . By Theorem 1.2 it suffices to bound  $d_{2,x}(t)$ .

Step 1: Show that (Proposition 3.3)

$$\forall B \in \text{Con}_{1/2}, \quad \mathbb{P}_x[T_{B^c} > t] \leq \pi(B)^3.$$

Step 2: Show that (Lemma 4.2) for  $A_s := \{y : h_t(x, y) \geq (s+1)\}$

$$\forall M \geq 1 \quad \|\mathbb{P}_x^t - \pi\|_{2,\pi}^2 \leq M^2 + \int_M^\infty 2s\pi(A_s)ds.$$

$\implies$  By Poincaré ineq. (4.1) it suffices that  $s\pi(A_s) \leq 2s^{-3/2}$  for  $s \geq M$  (for some  $M$ ).

Step 3: For  $B_s = \{y : \sup_k H_k(y, A_s) > \frac{s}{2}\pi(A_s)\}$  by step 1 and the Markov property,

$$\begin{aligned} s\pi(A_s) &\leq H_t(x, A_s) = \mathbb{P}_x[T_{B_s^c} > t, X_t \in A_s] + \mathbb{P}_x[T_{B_s^c} \leq t, X_t \in A_s] \\ \mathbb{P}_x[T_{B_s^c} > t] + \sup_{y \notin B_s, k \geq 0} H_k(y, A_s) &\leq \pi(B_s)^3 + \frac{s}{2}\pi(A_s). \end{aligned} \quad (1.19)$$

Step 4: If  $\pi(B_s) \leq s^{-1/2}$ , then we are done. Unfortunately, we do not know how to prove this estimate. Hence we have to define the set  $B_s$  in a slightly different manner:  $B_s := \{y : \sup_k H_k(y, A_s) > e\sqrt{s}|\log \pi(A_s)|\pi(A_s)\}$ . By Lemma 2.2 indeed  $\pi(B_s) \leq s^{-1/2}$ . Since  $e\sqrt{s}|\log \pi(A_s)|\pi(A_s) \leq s\pi(A_s)/2$ , unless  $\pi(A_s) \leq Ce^{-\sqrt{s}}$ , repeating the reasoning in (1.19) with the new choice of  $B_s$  concludes the proof.  $\square$



The proof of Theorem 1.2 is similar. The general scheme is as follows. Define a relevant family of sets  $A_s$ . Define  $B_s$  to be of the following form  $\{y : \sup |g_s(y)| > a_s\}$  with appropriate choices of  $g_s$  and  $a_s \in \mathbb{R}_+$  so that the desired inequality we wish to establish for  $A_s$  holds with some room to spare given that  $T_{B_s^c} \leq t$  (for an appropriate choice of  $t$ ). Finally, control the error term  $P[T_{B_s^c} > t]$  (using the choice of  $t$ ) by controlling  $\pi(B_s)$  using an appropriate maximal inequality.

## 1.8 Related work

Consider

$$\text{hit}(\varepsilon) := \max_x \text{hit}_x(\varepsilon), \quad \text{hit}_x(\varepsilon) := \min\{t : P_x[T_A > t] \leq \varepsilon, \forall A \in \text{Con}_{1/2}\}.$$

Let  $t_{\text{mix}}(\varepsilon) := \tau_1(2\varepsilon)$  be the total-variation  $\varepsilon$ -mixing-time. In [2] it was shown that for finite irreducible reversible chains, for all  $\varepsilon \in (0, 1)$ ,  $\delta \in (0, \frac{1}{2} \min(\varepsilon, 1 - \varepsilon))$  we have that

$$|\text{hit}(\varepsilon + \delta) - 4t_{\text{rel}} \log \delta| \leq t_{\text{mix}}(\varepsilon) \leq |\text{hit}(\varepsilon - \delta) + 4t_{\text{rel}} \log \delta|, \quad (1.20)$$

Generally,  $t_{\text{rel}} |\log(2\varepsilon)| \leq t_{\text{mix}}(\varepsilon)$  for all  $0 < \varepsilon \leq 1/2$ , however often  $t_{\text{rel}} \ll \tau_1$ . In particular, this is the case for a sequence of reversible chains which exhibits cutoff (i.e. abrupt convergence) in total variation ([12, Lemma 18.4]). In [2, Theorem 3] (1.20) is exploited in order to obtain a characterization of the cutoff phenomenon for reversible Markov chains, in terms of concentration of hitting times of “worst” (in some sense) sets.

The main tool in the proof of (1.20) is Starr’s  $L_p$  maximal inequality [17] (see Theorem 2.1 below). In other words, an  $L_p$  maximal inequality is used to characterize convergence in  $L_1$ . A look into the proof of (1.20) reveals that it does not require the full strength of Starr’s inequality (as in the proof it is applied only to functions taking values in  $[0, 1]$ ). It is thus natural to try applying Starr’s  $L_p$  maximal inequality to study stronger notions of convergence. Indeed Theorem 1.1 can be seen as the  $p > 1$  counterpart of (1.20). Also in our analysis the main tool is Starr’s inequality.

## 2 Starr maximal inequality and a useful lemma

In this section we prove a maximal inequality which shall be central in what comes. Denote  $S_t := e^{-(I-Q)t} = \sum_{k=0}^{\infty} \frac{e^{-t} t^k}{k!} Q^k$ . When considering  $Q$  instead of  $P$  we write  $\mathbb{P}_x^t$ ,  $\mathbb{P}_x$  and  $Y_t$  instead of  $P_x^t$ ,  $P_x$  and  $X_t$ , respectively.

**Theorem 2.1** (Starr’s Maximal inequality [17]). *Let  $(\Omega, P, \pi)$  be an irreducible Markov chain. Let  $f \in \mathbb{R}^\Omega$ . Its corresponding maximal function  $f^* \in \mathbb{R}^\Omega$  is defined as*

$$f^*(x) := \sup_{0 \leq t < \infty} |S_t(f)(x)| = \sup_{0 \leq t < \infty} |\mathbb{E}_x[f(Y_t)]|.$$

Then for every  $1 < p < \infty$

$$\|f^*\|_p \leq p^* \|f\|_p, \quad \text{where } p^* := p/(p-1) \text{ is the conjugate exponent of } p. \quad (2.1)$$



The following Lemma is essentially due to Norris, Peres and Zhai [14].

**Lemma 2.2.** *Let  $(\Omega, P, \pi)$  be a finite irreducible Markov chain. Let  $f_A(x) := 1_{x \in A}/\pi(A)$ .*

$$\forall A \subset \Omega, \quad \|f_A^*\|_1 \leq e \max(1, |\log \pi(A)|).$$

*Proof.* By (2.1) for all  $1 < p < \infty$

$$\|f_A^*\|_1 \leq \|f_A^*\|_p \leq p^* \|f_A\|_p = p^* [\pi(A)]^{-1/p^*},$$

Taking  $p^* := \max(1 + \varepsilon, |\log \pi(A)|)$  and sending  $\varepsilon$  to 0 (noting that the r.h.s. is continuous w.r.t.  $p^*$ ) concludes the proof.  $\square$

We note that by [17, Theorem 2]  $(1 - e^{-1})\|f_A^*\| - 1 \leq \|f_A \log[\max(1, |f_A|)]\|_1 = |\log \pi(A)|$ .

### 3 Bounding escape probabilities using $\kappa$

Recall that  $P_A$  and  $Q_A$  are the restriction to  $A$  of  $P$  and  $Q$ , resp.. Denote

$$H_t^A(x, y) := e^{-t(I-P_A)}(x, y) = \mathbb{P}_x(X_t = y, T_{A^c} > t) \quad \text{and similarly} \quad S_t^A := e^{-t(I-Q_A)}.$$

Recall that  $\lambda(A)$  is the smallest eigenvalue of  $I - Q_A$ . By the Perron-Frobenius Theorem there exists a distribution  $\mu_A$  on  $A$ , known as the **quasi-stationary distribution** of  $A$ , satisfying that the escape time from  $A$  w.r.t.  $Q$ , starting from  $\mu_A$ , has an Exponential (resp. Geometric in discrete-time) distribution with mean  $t_{\text{rel}}(A) = 1/\lambda(A)$ . Equivalently, for all  $t \geq 0$

$$\mu_A Q_A = (1 - \lambda(A))\mu_A \quad \text{and} \quad \mu_A S_t^A = e^{-\lambda(A)t} \mu_A.$$

Throughout we use  $\mu_A$  to denote the quasi-stationary distribution of  $A$ . Recall that we denote  $\pi$  conditioned on  $A$  by  $\pi_A$ .

Using the spectral decomposition of  $Q_A$  (e.g. [2, Lemma 3.8] or [1, (3.87)]) it follows that

$$\forall A \subsetneq \Omega, s \geq 0, \quad \mathbb{P}_{\pi_A}[T_{A^c} > s] \leq \mathbb{P}_{\mu_A}[T_{A^c} > s] = \mu_A S_s^A 1_A = e^{-\lambda(A)s} \mu_A(A) = e^{-\lambda(A)s}. \quad (3.1)$$

**Proposition 3.1.** *For reversible chains*

$$\kappa \leq 3\rho. \quad (3.2)$$

*Proof:* Let  $A \in \text{Con}_{1/2}$  be such that  $\kappa = t_{\text{rel}}(A)|\log \pi(A)|$ . By (3.1)  $\mathbb{P}_{\mu_A}[T_{A^c} > \kappa/3] = \pi(A)^{1/3}$ . Since  $a^{1/3} \geq a + \frac{1}{2}\sqrt{a(1-a)}$ , for all  $0 \leq a \leq 1/2$ , we get that

$$\max_{x \in A} \mathbb{P}_x[T_{A^c} > \kappa/3] \geq \mathbb{P}_{\mu_A}[T_{A^c} > \kappa/3] = \pi(A)^{1/3} \geq \pi(A) + \frac{1}{2}\sqrt{\pi(A)\pi(A^c)}. \quad \square$$

**Definition 3.2.**  $\bar{\rho} := \max_x \bar{\rho}_x$  and  $\bar{\rho}_{\text{Ent}} := \max_x \rho_{\text{Ent},x}$ , where

$$\begin{aligned} \bar{\rho}_x &:= \min\{t : \mathbb{P}_x[T_{A^c} > t] \leq \pi(A)^3 \text{ for all } A \in \text{Con}_{1/2}\}. \\ \bar{\rho}_{\text{Ent},x} &:= \min\{t : \mathbb{P}_x[T_{A^c} > t] \leq \frac{1}{16e^2[\log(e^{3/2}/\pi(A))]^3} \text{ for all } A \in \text{Con}_{1/2}\} \end{aligned} \quad (3.3)$$

Note that by the Markov property,  $\max_x P_x[T_{A^c} > mt] \leq (\max_y P_y[T_{A^c} > t])^m$  and so

$$\rho \leq \bar{\rho} \leq 8\rho \quad \text{and} \quad \rho_{\text{Ent}} \leq \bar{\rho}_{\text{Ent}} \leq C' \rho_{\text{Ent}}, \quad (3.4)$$

for some absolute constant  $C' > 0$ . The following proposition refines the inequality  $\bar{\rho} \leq 8\rho$ .

**Proposition 3.3.** *For every reversible chain,*

$$\forall x \in \Omega, \quad \bar{\rho}_x \leq \rho_x + s, \quad \text{where} \quad s := 8\kappa + 2t_{\text{rel}} \log 8. \quad (3.5)$$

*Proof:* Let  $x \in \Omega$  and  $A \in \text{Con}_{1/2}$ . By (1.6)  $2t_{\text{rel}} \geq \max_{B \in \text{Con}_{1/2}} t_{\text{rel}}(B)$  and so by (3.1)

$$P_{\pi_A}[T_{A^c} > s] \leq e^{-\lambda(A)[t_{\text{rel}}(A)(8|\log \pi(A)| + \log 8)]} = \pi(A)^8/8.$$

Thus the set

$$B = B(A) := \{y : P_y[T_{A^c} > s] > \pi(A)^3/2\}$$

satisfies

$$\pi(B)/\pi(A) = \pi_A(B) < P_{\pi_A}[T_{A^c} > s]/(\pi(A)^3/2) \leq \pi(A)^5/4,$$

and so by the definition of  $\rho_x$ ,  $P_x[T_{B^c} > \rho_x] \leq \pi(B) + \frac{1}{2}\sqrt{\pi(B)\pi(B^c)} \leq \sqrt{\pi(B)} \leq \frac{1}{2}\pi(A)^3$  (where we used  $\pi(B) < 2^{-8}$ ). Finally, by the definition of  $B$  and the Markov property

$$P_x[T_{A^c} > \rho_x + s] \leq P_x[T_{B^c} > \rho_x] + \max_{b \notin B} P_b[T_{A^c} > s] \leq \frac{1}{2}\pi(A)^3 + \frac{1}{2}\pi(A)^3 = \pi(A)^3. \quad \square$$

## 4 An upper bound on $\tau_2$

In this section we prove the following theorem.

**Theorem 4.1.** *For every finite irreducible reversible Markov chain  $(\Omega, P, \pi)$  we have that*

$$\forall x, \quad \rho_x \leq \tau_{2,x} \leq \bar{\rho}_x + 4et_{\text{rel}} \leq \rho_x + 8\kappa + (4e + 6 \log 2)t_{\text{rel}}. \quad (4.1)$$

*The same holds when  $x$  is omitted from all of the terms above. Consequently,*

$$\rho \leq \tau_2 \leq (8 + 12e/\log 2)\rho. \quad (4.2)$$

The following fact (often referred to as the Poincaré inequality) is standard. It can be proved by elementary linear-algebra using the spectral decomposition (e.g. [1, Lemma 3.26]).

**Fact 4.1.** *Let  $(\Omega, P, \pi)$  be a finite irreducible Markov chain. Let  $x \in \Omega$  and  $s, t \geq 0$ . Then*

$$\|P_x^{t+s} - \pi\|_{2,\pi} \leq e^{-s/t_{\text{rel}}} \|P_x^t - \pi\|_{2,\pi}. \quad (4.3)$$

*In particular, for all  $x \in \Omega$  and  $M \geq 1$ ,*

$$\tau_{2,x} \leq \tau_{2,x}(M/2) + t_{\text{rel}} \log M.$$

**Lemma 4.2.** Let  $A_{x,t}(s) := \{y : h_t(x, y) \geq s + 1\}$ . For every finite irreducible reversible chain, for all  $x \in \Omega$  and  $\ell \geq 1$

$$\forall t \geq 0, \quad \|P_x^t - \pi\|_{2,\pi}^2 \leq \ell^2 + \int_{\ell}^{\infty} 2s\pi(A_{x,t}(s))ds.$$

*Proof:* Fix some  $x \in \Omega$ ,  $t \geq 0$  and  $\ell \geq 1$ . Let  $f(y) := |h_t(x, y) - 1|$ . Then  $\|P_x^t - \pi\|_{2,\pi}^2 = \|f\|_2^2 = \mathbb{E}_{\pi}[f^2]$ . Note that for all  $s > 1$ ,  $\{f \geq s\} = A_{x,t}(s)$ . Observe that

$$\mathbb{E}_{\pi}[f^2 1_{f>\ell}] = \int_0^{\infty} 2s\pi(\{f 1_{f>\ell} > s\})ds \leq \pi(f > \ell)\ell^2 + \int_{\ell}^{\infty} 2s\pi(A_{x,t}(s))ds.$$

Finally, since  $f^2 \leq f^2 1_{f>\ell} + 1_{f \leq \ell}\ell^2$ , we get that

$$\mathbb{E}_{\pi}[f^2] \leq \pi(f \leq \ell)\ell^2 + \mathbb{E}_{\pi}[f^2 1_{f>\ell}] \leq \ell^2 + \int_{\ell}^{\infty} 2s\pi(A_{x,t}(s))ds \quad \square.$$

*Proof of Theorem 4.1:* Let  $x \in \Omega$ . The inequality  $\rho_x \leq \tau_{2,x}$  follows from (1.18). Set  $t := \bar{\rho}_x$ . As above, denote  $A_s := \{y : h_t(x, y) \geq s + 1\}$ . By Fact 4.1 and Lemma 4.2 it suffices to show that

$$\int_{e^e}^{\infty} 2s\pi(A_s)ds \leq e^{8e}/4 - e^{2e}.$$

Let  $g_s(y) := \sup_k H_k(y, A_s)/\pi(A_s)$ . By Lemma 2.2  $\|g\|_1 \leq e|\log \pi(A_s)|$ . Let

$$B_s := \{y : g_s(y) > e\sqrt{s+1}|\log \pi(A_s)|\} = \{y : \sup_k H_k(y, A_s) \geq e\sqrt{s+1}\pi(A_s)|\log \pi(A_s)|\}.$$

Let  $s \geq e^e$ . By Markov inequality  $\pi(B_s) \leq 1/\sqrt{s+1} \leq \frac{1}{2}$  and so by the definition of  $\bar{\rho}_x$

$$\mathbb{P}_x[T_{B_s^c} > t, X_t \in A_s] \leq \mathbb{P}_x[T_{B_s^c} > t] \leq \frac{1}{(s+1)^{3/2}}.$$

Also, by the definition of  $B_s$  we clearly have that

$$\mathbb{P}_x[T_{B_s^c} \leq t, X_t \in A_s] \leq \sup_{b \notin B_s, k \geq 0} H_k(b, A_s) \leq e\sqrt{s+1}\pi(A_s)|\log \pi(A_s)|.$$

Since by the definition of  $A_s$

$$(s+1)\pi(A_s) \leq H_t(x, A_s) = \mathbb{P}_x[T_{B_s^c} > t, X_t \in A_s] + \mathbb{P}_x[T_{B_s^c} \leq t, X_t \in A_s],$$

we get that if  $\mathbb{P}_x[T_{B_s^c} > t, X_t \in A_s] \leq \mathbb{P}_x[T_{B_s^c} \leq t, X_t \in A_s]$ , then

$$(s+1)\pi(A_s) \leq 2e\sqrt{s+1}\pi(A_s)|\log \pi(A_s)|,$$

which simplifies as follows

$$2s\pi(A_s) \leq 2se^{-\sqrt{s+1}+2e}.$$

while if  $\mathbb{P}_x[T_{B_s^c} > t, X_t \in A_s] > \mathbb{P}_x[T_{B_s^c} \leq t, X_t \in A_s]$ , then we have that

$$2s\pi(A_s) < 4\mathbb{P}_x[T_{B_s^c} \leq t, X_t \in A_s] \leq \frac{4}{(s+1)^{3/2}}.$$

In conclusion, as desired,

$$\int_{e^e}^{\infty} 2s\pi(A_s)ds \leq \int_{e^e}^{\infty} \max(2se^{-\sqrt{s+1}+2e}, \frac{4}{(s+1)^{3/2}})ds \leq e^{8e}/4 - e^{2e}. \quad \square$$

## 5 A characterization of the Log-Sobolev constant

### 5.1 Background

There are numerous works aiming towards general geometric upper bounds on  $\tau_\infty$ . Among the most advanced techniques are the spectral profile [8] and Logarithmic Sobolev inequalities (see [5] for a survey on the topic). Let  $\pi_* := \min_{x \in \Omega} \pi(x)$ . It is classical (e.g. [5, Corollary 3.11]) that for reversible chains

$$t_{\text{LS}}/2 \leq \tau_2(1/e) \leq t_{\text{LS}}(1 + \frac{1}{4} \log \log(1/\pi_*)). \quad (5.1)$$

There are examples demonstrating that each of these bounds can be attained up to a constant factor.

### 5.2 Hypercontractivity

We start by recalling the notion of hypercontractivity and its connection with the log-sobolev constant. Let  $1 \leq p_1, p_2 \leq \infty$ . The  $p_1 \rightarrow p_2$  norms of a linear operator  $\mathbf{A}$  are given by

$$\|\mathbf{A}\|_{p_1 \rightarrow p_2} := \max\{\|\mathbf{A}f\|_{p_2} : \|f\|_{p_1} = 1\}.$$

If  $\|\mathbf{A}\|_{p_1 \rightarrow p_2} \leq 1$  for some  $1 \leq p_2 < p_1 \leq \infty$  we say that  $\mathbf{A}$  is a hypercontraction. For all  $p_1, p_2$ ,  $\|H_t\|_{p_1 \rightarrow p_2}$  is non-increasing in  $t$ . It is a classic result (e.g. [5, Theorem 3.5] and [1, Theorem 8.24]) that the Log-Sobolev time can be characterized in terms of hypercontractivity.

**Fact 5.1.** *Let  $(\Omega, P, \pi)$  be a finite reversible chain. Let  $s_q := \inf\{t : \|H_t\|_{2 \rightarrow q} \leq 1\}$ . Then  $t_{\text{LS}} = 4 \sup_{q: 2 < q < \infty} s_q / \log(q - 1)$ .*

The following result ([5, Theorem 3.10]) will allow us to bound  $t_{\text{LS}}$  from above.

**Fact 5.2.** *Let  $(\Omega, P, \pi)$  be a finite reversible chain. Fix  $2 < q < \infty$ . Assume that  $r_q$  and  $M_q$  satisfy that  $\|H_{r_q}\|_{2 \rightarrow q} \leq M_q$ . Then*

$$t_{\text{LS}} \leq \frac{2q}{q-2} r_q + 2t_{\text{rel}}(1 + \frac{q}{q-2} \log M_q). \quad (5.2)$$

Fix some  $0 < \varepsilon < 1/2$  and  $A \in \text{Con}_{2^{-1/\varepsilon}}$ . Assume that  $\mathbb{P}_\pi[T_{A^c} > t] \geq 2\pi(A)^{1+\varepsilon}$ . Recall that  $\pi_A$  denotes  $\pi$  conditioned on  $A$ . Then  $\mathbb{P}_{\pi_A}[T_{A^c} > t] \geq 2\pi(A)^\varepsilon$  and so

$$B = \{a \in A : \mathbb{P}_a[T_{A^c} > t] \geq \pi(A)^\varepsilon\}$$

satisfies  $\pi_A(B) \geq \pi(A)^\varepsilon$  (i.e.  $\pi(B) \geq \pi(A)^{1+\varepsilon}$ ). Consequently, for  $q > \frac{2(1+\varepsilon)}{1-2\varepsilon}$

$$\|H_t 1_A\|_q \geq \left[ \sum_{b \in B} \pi(b) H_t(b, A)^q \right]^{1/q} \geq \pi(B)^{1/q} \pi(A)^\varepsilon \geq \pi(A)^{\varepsilon + (1+\varepsilon)/q} > \sqrt{\pi(A)} = \|1_A\|_2.$$

Thus a natural hitting time version of hypercontractivity is

$$t_{\text{ht}} := \min\{t : \mathbb{P}_\pi[T_{A^c} > t] \leq \pi(A)^{5/4} \text{ for all } A \in \text{Con}_{1/2}\}.$$

**Question.** *There exists an absolute constant  $C$  such that for every finite irreducible reversible Markov chain  $t_{\text{ht}}/C \leq t_{\text{LS}} \leq Ct_{\text{ht}}$ .*

Trivially,  $t_{\text{ht}} = \min\{t : P_{\pi_A}[T_{A^c} > t] \leq \pi(A)^{1/4} \text{ for all } A \in \text{Con}_{1/2}\}$ . Note that if we replace  $\pi_A$  by the quasi-stationary distribution of  $A$ ,  $\mu_A$ , then by (3.1) we get precisely  $\kappa/4$ . This explains why also  $\kappa$  can be interpreted as a hitting time version of hypercontractivity. We note that Conjecture 5.2 resembles Open problem 4.38 in [1], which asks whether for reversible chains  $t_{\text{rel}} \leq C \max_{A \in \text{Con}_{1/2}} \mathbb{E}_{\pi_A}[T_{A^c}]$ , where indeed [1, Lemma 4.39]  $t_{\text{rel}} \leq \max_{A \in \text{Con}_{1/2}} \mathbb{E}_{\mu_A}[T_{A^c}]$  (the formulation in [1] is slightly different, but it is equivalent to our formulation).

### 5.3 Proof of Theorem 1.2

*Proof of Theorem 1.2:* As mentioned in the introduction, it is known that  $\kappa \leq t_{\text{LS}}$ . Denote  $r := \frac{1}{2}\kappa$ . Note that  $P$  and  $Q = (P + P^*)/2$  have the same  $t_{\text{rel}}$  and  $t_{\text{LS}}$ . Thus we may work with  $S_t = e^{-t(I-Q)}$  instead of  $H_t$ . By (5.2) it suffices to show that  $\|S_r\|_{2 \rightarrow 4} \leq 7$ . Fix some  $f \in \mathbb{R}^\Omega$  such that  $\|f\|_2 = 1$ . Our goal is to show that  $\|S_r f\|_4 \leq 7$ . By considering  $|f|$  instead of  $f$  we may assume that  $f \geq 0$ . Let

$$A_s := \{x : S_r f(x) \geq s\}.$$

Then  $\|S_r f\|_4^4 = \int_0^\infty 4s^3 \pi(A_s) ds \leq 6^4 + \int_6^\infty 4s^3 \pi(A_s) ds$ . Hence to conclude the proof

$$\text{it suffices to show that } \int_6^\infty 4s^3 \pi(A_s) ds \leq 16 \leq 7^4 - 6^4. \quad (5.3)$$

Recall that  $S_t f(x) = \mathbb{E}_x[f(Y_t)]$  and that for all  $A \subset \Omega$ ,  $S_t^A f(a) = \mathbb{E}_a[f(Y_t)1_{T_{A^c} > t}]$ . Let

$$B_s := \{x : \sup_t S_t f(x) > s/2\} = \{f^* > s/2\}, \quad \text{where } f^*(x) = \sup_t S_t f(x)$$

$$D_s := \{x \in B_s : \mathbb{E}_x[f(Y_r)1_{T_{B_s^c} > r}] \geq s/2\}, \quad F_s := \{x \in B_s : \mathbb{E}_x[f^2(Y_r)1_{T_{B_s^c} > r}] \geq s^2/4\}.$$

By the Markov property (first inclusion),  $A_s \subset D_s \subset F_s$  (the second inclusion follows by the Cauchy-Schwarz inequality). Thus  $\pi(A_s) \leq \pi(F_s)$ . Hence, by (5.3) in order to conclude the proof it suffices to show that  $\int_6^\infty 4s^3 \pi(F_s) ds \leq 16$ . By Starr's maximal inequality (2.1) we know that  $\int_0^\infty 4s \pi(B_s) ds = \|f^*\|_2^2 \leq 4\|f\|_2^2 = 4$ . Thus in order to show that  $\int_6^\infty 4s^3 \pi(F_s) ds \leq 16$ , and conclude the proof, it suffices to show that for all  $s \geq 6$  we have that  $\pi(F_s) \leq 4s^{-2} \pi(B_s)$ .

Fix some  $s \geq 6$ . Note that since  $\|f^*\|_2^2 \leq 4$ , by Markov inequality we have that  $\pi(B_s) \leq 16/s^2 < 1/2$ . Using the spectral decomposition of the restriction of  $f$  to  $B_s$  (c.f. [2, Lemma 3.8]) and the choice of  $r$

$$\mathbb{E}_{\pi_{B_s}}[f^2(Y_r)1_{T_{B_s^c} > r}] \leq \mathbb{E}_{\pi_{B_s}}[f^2(Y_0)]e^{-2\lambda(B_s)r} \leq (\|f\|_2^2/\pi(B_s))e^{-2\lambda(B_s)r} = (1/\pi(B_s)) \times \pi(B_s) = 1.$$

Thus by the def. of  $F_s$ ,  $\frac{1}{4}s^2 \pi_{B_s}(F_s) \leq \sum_{y \in F_s} \pi_{B_s}(y) \mathbb{E}_y[f^2(Y_r)1_{T_{B_s^c} > r}] \leq \mathbb{E}_{\pi_{B_s}}[f^2(Y_r)1_{T_{B_s^c} > r}] \leq 1$  and so indeed  $\pi(F_s) \leq 4s^{-2} \pi(B_s)$ .  $\square$

## 6 A hitting times characterization of mixing in relative entropy

### 6.1 Background

The relative entropy distance can be compared with the  $L_1$  and  $L_2$  distances as follows: [11]  $2D(\mu||\pi) \geq \|\mu - \pi\|_{1,\pi}^2$  and ([7, Theorem 5])

$$D(\mu||\pi) \leq \log(1 + \|\mu - \pi\|_{2,\pi}^2). \quad (6.1)$$

Recall the definitions of  $\rho_{\text{Ent}}, \bar{\rho}_{\text{Ent}}, \rho_{\text{Ent},x}$  and  $\bar{\rho}_{\text{Ent},x}$  from (1.1) and (3.3). Recall that by (3.4),  $\rho_{\text{Ent}} \leq \bar{\rho}_{\text{Ent}} \leq C\rho_{\text{Ent}}$ . The following theorem refines (1.4) from Theorem 1.1.

**Theorem 6.1.** *Let  $(\Omega, P, \pi)$  be a finite irreducible reversible Markov chain. Then*

$$\forall x, \quad \rho_{x,\text{Ent}} \leq \tau_{\text{Ent},x} \leq \bar{\rho}_{x,\text{Ent}} + 14t_{\text{rel}}. \quad (6.2)$$

*The same holds when  $x$  is omitted from all of the terms above. Consequently*

$$\rho_{\text{Ent}} \leq \tau_{\text{Ent}} \leq C_1\rho_{\text{Ent}}. \quad (6.3)$$

### 6.2 Proof of Theorem 6.1

*Proof of Theorem 6.1:* Let  $x \in \Omega$ . The inequality  $\rho_{x,\text{Ent}} \leq \tau_{\text{Ent},x}$  follows from (1.18). The inequality  $\tau_{\text{Ent}} \leq C_1\rho_{\text{Ent}}$  follows from (6.2) and (3.4), in conjunction with the fact that (under reversibility)  $ct_{\text{rel}} \leq \rho_{\text{Ent}}$  for some absolute constant  $c > 0$  (c.f. [2, (3.19)] for the fact that there exist some  $A \in \text{Con}_{1/2}$  and  $a \in A$  so that  $P_a[T_{A^c} > \varepsilon t_{\text{rel}}] \geq e^{-\varepsilon} \geq 1 - \varepsilon$ , for all  $\varepsilon \geq 0$ ). We now prove that  $\tau_{\text{Ent},x} \leq \bar{\rho}_{x,\text{Ent}} + 14t_{\text{rel}}$ . Denote  $r := \bar{\rho}_{x,\text{Ent}}, r' := 14t_{\text{rel}}$ . Let

$$D := \{y : h_r(x, y) > e^{10}\}.$$

Denote  $\delta := H_r(x, D) - e^{10}\pi(D)$ ,

$$\mu(y) := \delta^{-1}1_{y \in D}[H_r(x, y) - e^{10}\pi(y)],$$

$$\nu(y) := (1 - \delta)^{-1}[1_{y \notin D}H_r(x, y) + 1_{y \in D}e^{10}\pi(y)].$$

Denote  $\mu_\ell := \mu H_\ell$  and  $\nu_\ell := \nu H_\ell$ . Then  $P_x^{r+r'} = \delta\mu_{r'} + (1 - \delta)\nu_{r'}$  and so by the triangle inequality (which holds for  $D$ , by Jensen's inequality applied to each  $y$  separately) and (6.1)

$$D(P_x^{r+r'}||\pi) \leq \delta D(\mu_{r'}||\pi) + (1 - \delta)D(\nu_{r'}||\pi) \leq \delta D(\mu_{r'}||\pi) + (1 - \delta)\log(1 + \|\nu_{r'} - \pi\|_{2,\pi}^2). \quad (6.4)$$

By (4.3)

$$\|\nu_{r'} - \pi\|_{2,\pi} \leq \|\nu - \pi\|_{2,\pi}e^{-14} \leq \|\nu - \pi\|_{\infty,\pi}e^{-14} \leq (1 - \delta)^{-1}e^{-4}.$$

Using  $\sqrt{1+a} \leq 1 + \sqrt{a}$  and  $\log(1+a) \leq a$  we get that

$$(1 - \delta)\log(1 + \|\nu_{r'} - \pi\|_{2,\pi}^2) \leq 2(1 - \delta)\log(1 + \|\nu_{r'} - \pi\|_{2,\pi}) \leq 2e^{-4}.$$

By (6.4) to conclude the proof it is left to show that  $\delta D(\mu_{r'}||\pi) \leq 1/2 - 2e^{-4}$ . Denote

$$a_y := 1_{y \in D}[H_r(x, y) - e^{10}\pi(y)], \quad g(y) = a_y/\pi(y).$$

$$\delta D(\mu_{r'}||\pi) = \sum a_y \log(g(y)/\delta) = \delta |\log \delta| + \mathbb{E}_\pi[g \log g].$$

Since  $\delta |\log \delta| \leq 1/e$ , for all  $\delta \in [0, 1]$ , in order to show that  $\delta D(\mu_{r'}||\pi) \leq 1/2 - 2e^{-4}$

$$\text{it suffices to show that } \mathbb{E}_\pi[g \log g] \leq 1/10 < 1/2 - 1/e - 2e^{-4}. \quad (6.5)$$

Similarly to the proof of Theorem 4.1, let

$$A_s = \{y : g(y) \geq s\} \text{ and } B_s := \{y : \sup_{\ell} H_{\ell}(y, A_s) > \sqrt{s + e^{10}} \pi(A_s) |\log \pi(A_s)|\}.$$

Then

$$\mathbb{E}_\pi[g \log g] \leq \int_0^\infty \pi(\{y : g(y) \log g(y) > s\}) ds = \int_1^\infty (1 + \log s) \pi(A_s) ds. \quad (6.6)$$

Note that  $(e^{10} + s)\pi(y) \leq H_r(x, y)$  for every  $y \in A_s$ . Hence as in the proof of Theorem 4.1

$$(e^{10} + s)\pi(A_s) \leq H_r(x, A_s) \leq \mathbb{P}_x[T_{B_s^c} > r] + \mathbb{E}_x[X_r \in A_s \mid T_{B_s^c} \leq t]. \quad (6.7)$$

By the definition of  $B_s$  and the Markov property,

$$\mathbb{E}_x[X_r \in A_s \mid T_{B_s^c} \leq r] \leq \sup_{y \notin B_s, \ell \geq 0} H_{\ell}(y, A_s) \leq \sqrt{s + e^{10}} \pi(A_s) |\log \pi(A_s)|. \quad (6.8)$$

By Lemma 2.2  $\pi(B_s) \leq e/\sqrt{s + e^{10}} \leq 1/2$  and hence by the definition of  $r$ ,

$$\mathbb{P}_x[T_{B_s^c} > r] \leq \frac{1}{16e^2(\frac{1}{2}(\log(s + e^{10}) + 1))^3} = \frac{1}{2e^2(1 + \log(s + e^{10}))^3}.$$

As in the proof of Theorem 4.1, it follows that for all  $s \geq 1$ ,  $(s + e^{10})\pi(A_s) \leq \frac{2}{2e^2(1 + \log(s + e^{10}))^3}$ , as otherwise by (6.7)  $(s + e^{10})\pi(A_s) < 2\mathbb{E}_x[X_r \in A_s \mid T_{B_s^c} \leq t]$ , which by (6.8) implies that

$$\pi(A_s) \leq \exp(-\frac{1}{2}\sqrt{s + e^{10}}) \leq \exp(-\sqrt{s/8} + \sqrt{e^{10}/8}) < e^{-50 - \sqrt{s/8}} < \frac{(s + e^{10})^{-1}}{e^2(1 + \log(s + e^{10}))^3},$$

a contradiction. Thus for all  $s \geq 1$ ,

$$(1 + \log s)\pi(A_s) \leq \frac{1}{e^2(s + e^{10})(1 + \log(s + e^{10}))^2},$$

which yields that  $\int_1^\infty (1 + \log s)\pi(A_s) ds \leq \int_{1+e^{10}}^\infty \frac{e^{-2} ds}{s(1 + \log s)^2} = \frac{e^{-2}}{1 + \log(1 + e^{10})} < e^{-2}/11$ . This concludes the proof using (6.5) and (6.6).  $\square$

## 7 Application to robustness of mixing

### 7.1 Proof of Corollary 1.3

*Proof.* It is not hard to verify that Theorem 1.1 is still valid in the above setup (this can be formally deduced from Theorem 1.1 via the representation of the generator appearing in the paragraph following Corollary 1.3). Hence it suffices to verify that (1.8) is valid if



we replace  $\tau_2$  and  $\tilde{\tau}_2$  by  $\rho$  and  $\tilde{\rho}$ , resp. (where  $\tilde{\rho}$  is the parameter  $\rho$  of the chain  $(\tilde{X}_t)$ ). A straightforward coupling of the chains in which they follow the same trajectory (i.e. they make the same sequence of jumps, possibly at different times) shows that for all  $x$  and  $A$  the hitting time of  $A$  starting from  $x$  for the two chains,  $T_A$  and  $\tilde{T}_A$ , resp., satisfy that  $\tilde{T}_A/M \leq_{\text{st}} T_A \leq_{\text{st}} M\tilde{T}_A$ , where  $\leq_{\text{st}}$  denotes stochastic domination. Since for all  $A$  we have that  $\tilde{\pi}(A)/M \leq \pi(A) \leq M\tilde{\pi}(A)$ , by the submultiplicity property

$$\forall t \geq 0, m \in \mathbb{Z}_+ \text{ and } A \subset \Omega, \quad \max_x \mathbb{P}_x[T_A > tm] \leq (\max_x \mathbb{P}_x[T_A > t])^m,$$

this implies that  $\tilde{\rho}/(C_1 M \log M) \leq \rho \leq (C_1 M \log M) \tilde{\rho}$ , as desired.  $\square$

## 7.2 Robustness of trees

We start with a few definitions. Given a network  $(V, E, (c_e)_{e \in E})$ , where each edge  $\{u, v\} \in E$  is endowed with a conductance (weight)  $c_{u,v} = c_{v,u} > 0$ , a random walk on  $(V, E, (c_e)_{e \in E})$  repeatedly does the following: when the current state is  $v \in V$ , the random walk will move to vertex  $u$  (such that  $\{u, v\} \in E$ ) with probability  $c_{u,v}/c_v$ , where  $c_v := \sum_{w: \{v,w\} \in E} c_{v,w}$ . This is a reversible Markov chain whose stationary distribution is given by  $\pi(x) := c_x/c_V$ , where  $c_V := \sum_{v \in V} c_v = 2 \sum_{e \in E} c_e$ . Conversely, every reversible Markov chain can be presented in this manner by setting  $c_{x,y} = \pi(x)P(x, y)$  (e.g. [12, Section 9.1]).

Let  $\mathcal{T} := (V, E)$  be a finite tree. By Kolmogorov's cycle condition every Markov chain on  $\mathcal{T}$  (i.e.  $P(x, y) > 0$  iff  $\{x, y\} \in E$ ) is reversible. Hence we may assume that  $\mathcal{T}$  is equipped with edge weights  $(c_e)_{e \in E}$ . Following [15], we call a vertex  $v \in V$  a central-vertex if each connected component of  $\mathcal{T} \setminus \{v\}$  has stationary probability at most  $1/2$ . A central-vertex always exists (and there may be at most two central-vertices). Throughout, we fix a central-vertex  $o$  and call it the *root* of the tree. The root induces a partial order  $\prec$  on  $V$ , as follows. For every  $u \in V$ , we denote the shortest path between  $u$  and  $o$  by  $\ell(u) = (u_0 = u, u_1, \dots, u_k = o)$ . We call  $u_1$  the *parent* of  $u$ . We say that  $u' \prec u$  if  $u' \in \ell(u)$  (i.e.  $u$  is a descendant of  $u'$  or  $u = u'$ ). The *induced tree* at  $u$  is  $\mathcal{T}_u := \{v : u \in \ell(v)\} = \{u\} \cup \{v : v \text{ is a descendant of } u\}$ . Fix some leaf  $x$  and  $\delta \in (0, 1/2)$ . Let  $W_{x,\delta}$  be the collection of all  $y \prec x$  such that  $\pi(\mathcal{T}_y) \geq \delta$  and let

$$x_\delta := \operatorname{argmin}\{\pi(\mathcal{T}_y) : y \in W_{x,\delta}\}$$

(i.e.  $d(x, x_\delta) = \min_{y \in W_{x,\delta}} d(x, y)$ , where  $d$  denotes the graph distance w.r.t.  $\mathcal{T}$ ). Recall that  $\alpha(A) = \lambda(A)/|\log \pi(A)|$  and that by Theorem 1.2,  $\alpha := \sup_{A \in \text{Con}_{1/2}} \alpha(A) \geq c_{\text{LS}}$ . Let  $D_\beta = D_{\beta,x}$  be the connected component of  $x$  in  $\mathcal{T} \setminus \{x_\beta\}$ . For a leaf  $x$  we denote

$$\alpha_x(\delta) := \alpha(D_\delta) \quad \text{and} \quad \alpha_x := \max_{\delta \in (0, 1/4]} \alpha_x(\delta) \geq \alpha.$$

Let us now describe the skeleton of the argument in the proof of Theorem 1.3.

Step 1 Show that it suffices to consider leafs as initial states. More precisely

**Lemma 7.1.** *There exists an absolute constant  $C > 0$  so that if  $y \prec x$  then*

$$\tau_{2,y} \leq \tau_{2,x} + C(t_{\text{LS}} + \sqrt{t_{\text{rel}}\tau_1}). \quad (7.1)$$

Step 2 Show that for a leaf  $x$  we can replace (in (3.4))  $\bar{\rho}_x$  (defined in (3.3)) with

$$b_x := \sup_{\delta \in (0, 1/4]} b_x(\delta) \quad \text{where} \quad b_x(\delta) := \min\{t : P_x[T_{x_\delta} > t] \leq \delta^3/4\}.$$

**Proposition 7.2.** *Let  $x$  be a leaf. Let  $0 < \delta \leq 1/4$  and  $A \in \text{Con}_\delta$ . Denote  $\bar{A} = A^c \setminus D_\delta$ , where  $D_\beta = D_{\beta, x}$  is the connected component of  $x$  in  $\mathcal{T} \setminus \{x_\beta\}$ . Then*

$$P_x[T_{A^c} > b_x + 3\kappa + 10t_{\text{rel}}] \leq P_x[T_{x_\delta} > b_x] + P_{x_\delta}[T_{\bar{A}} > 3\kappa + 10t_{\text{rel}}] < \delta^3/2. \quad (7.2)$$

Step 3 For a leaf  $x$  and  $\delta \in (0, 1/4]$ , derive a large deviation estimate for  $T_{x_\delta}$ :

**Proposition 7.3.** *There exists some  $C > 0$  so that for a leaf  $x$  and  $\delta \in (0, 1/4]$ ,*

$$b_x(\delta) \leq \mathbb{E}_x[T_{x_\delta}] + \max\left(\frac{32}{\alpha_x(\delta)}, 8\sqrt{\mathbb{E}_x[T_{x_\delta]}/\alpha_x(\delta)}\right) \leq \tau_1 + C \max(\kappa, \sqrt{\kappa\tau_1}). \quad (7.3)$$

The second inequality follows from the first using the fact that  $\mathbb{E}_x[T_{x_\delta}] \leq \tau_1 + C_5\sqrt{\tau_1 t_{\text{rel}}}$  [2, Corollary 5.5].

Step 4 Similar reasoning as in the proof of (3.5) yields that (c.f. [2, Corollary 3.4])

$$\bar{\rho}_x \leq \min\{t : P_x[T_{A^c} > t] \leq \pi(A)^3/2 \text{ for all } A \in \text{Con}_{1/4}\} + 10t_{\text{rel}}$$

By (7.1)-(7.3) in conjunction with (4.1) and (1.7) we have that

$$\begin{aligned} \tau_2 - C_1\sqrt{t_{\text{rel}}\tau_1} &\leq \max_{x:x \text{ a leaf}} \tau_{2,x} + C_1 t_{\text{LS}} \leq \max_{x:x \text{ a leaf}} \bar{\rho}_x + C_2 t_{\text{LS}} \\ &\leq \max_{x:x \text{ a leaf}} b_x + C_3 t_{\text{LS}} \leq \tau_1 + C_4 \max(t_{\text{LS}}, \sqrt{t_{\text{LS}}\tau_1}). \quad \square \end{aligned}$$

**Remark 7.4.** *While it is intuitive that “typically” the worst initial state is a leaf (i.e.  $\tau_2 = \tau_{2,x}$  for some leaf  $x$ ), it seems that this is not always the case. Consider a birth and death chain on  $\{1, \dots, 2n + \lceil n^{2/3} \rceil\}$  with a fixed bias to toward  $2n + \lceil n^{2/3} \rceil$ , apart from in the interval  $\{n, \dots, n + \lceil n^{2/3} \rceil\}$  in which it is unbiased. A-priori, it seems plausible that the worst initial state is  $n + \lceil n^{2/3}/2 \rceil$ .*

To conclude the proof of Theorem 1.3 we now prove Lemma 7.1 and Propositions 7.2-7.3.

*Proof of Lemma 7.1:* Let  $y \prec x$ . Let  $s := \tau_{2,y} - Mt_{\text{LS}}$  for some constant  $M > 0$  to be determined later. We may assume  $s > 16\sqrt{t_{\text{rel}}\tau_1}$  as otherwise there is nothing to prove. Form the proofs of (3.5) and (4.1) it follows that we can choose  $M$  so that for some  $A \in \text{Con}_{1/100}$

$$P_y[T_{A^c} > s] > 2\pi(A) + \sqrt{\pi(A)\pi(A^c)}. \quad (7.4)$$

We leave this as an exercise (the main issue is moving from an estimate for some  $B \in \text{Con}_{1/2}$  to one for some  $A \in \text{Con}_{1/100}$ . This can be done using similar reasoning as in the proof of (3.5), c.f. [2, Corollary 3.4]).

Denote the connected component of  $x$  in  $\mathcal{T} \setminus \{y\}$  by  $A$ . Since  $y \prec x$  we have  $\pi(A) \leq 1/2$ . Hence, for all  $z \in A$  we have  $P_z[T_y > \tau_1] \leq H_{\tau_1}(z, A) \leq \pi(A) + 1/4 \leq 3/4$ . Using the Markov property, by induction we get that  $P_z[T_y > k\tau_1] \leq (3/4)^k$  for all  $k \in \mathbb{N}$  and so  $\mathbb{E}_x[T_y] \leq 4\tau_1$ .

Let  $(v_0 = x, v_1, \dots, v_k = y)$  be the path from  $x$  to  $y$ . Define  $\xi_i := T_{v_i} - T_{v_{i-1}}$ . Then by the tree structure, under  $P_x$ , we have that  $T_y = \sum_{i=1}^k \xi_i$  and that  $\xi_1, \dots, \xi_k$  are independent. Denote  $\Phi(\mathcal{T}_{v_i}) := \frac{\pi(v_i)P(v_i, v_{i+1})}{\pi(\mathcal{T}_{v_i})}$ . By specializing Kac's formula to trees (see [1, (2.23)] for the general Kac's formula we are using and for its specialization for trees see (7.9) below and c.f. [2, Proposition 5.6 and Lemma 5.2]) we have that  $\mathbb{E}_{v_{i-1}}[T_{v_i}] = 1/\Phi(\mathcal{T}_{v_i})$  and that  $\mathbb{E}_{v_{i-1}}[T_{v_i}^2] \leq 2\mathbb{E}_{v_{i-1}}[T_{v_i}]\mathbb{E}_{\pi_{\mathcal{T}_{v_{i-1}}}}[T_{v_i}] \leq 4t_{\text{rel}}\mathbb{E}_{v_{i-1}}[T_{v_i}]$ . Whence,

$$\text{Var}_x[T_y] = \sum_{i=1}^k \text{Var}_{v_{i-1}}[T_{v_i}] \leq \sum_{i=1}^k \mathbb{E}_{v_{i-1}}[T_{v_i}^2] \leq 4t_{\text{rel}} \sum_{i=1}^k \mathbb{E}_{v_{i-1}}[T_{v_i}] = 4t_{\text{rel}}\mathbb{E}_x[T_y] \leq 16t_{\text{rel}}\tau_1.$$

By Chebyshev inequality

$$P_x[|T_y - \mathbb{E}_x[T_y]| > 8\sqrt{t_{\text{rel}}\tau_1}] \leq 1/4. \quad (7.5)$$

Let  $s' := \max(\mathbb{E}_x[T_y] - 8\sqrt{t_{\text{rel}}\tau_1}, 0)$ . By (7.4), (7.5),  $s > 16\sqrt{t_{\text{rel}}\tau_1}$  and the Markov property

$$P_x[X_{s+s'} \in A] \geq P_x[|T_y - \mathbb{E}_x[T_y]| \leq 8\sqrt{t_{\text{rel}}\tau_1}] \times P_y[T_{A^c} > s] > \pi(A) + \frac{1}{2}\sqrt{\pi(A)\pi(A^c)}.$$

The proof is concluded using (1.18) (in the notation from (1.18),  $P_x^{s+s'} \in \mathcal{P}_{A,\delta}$  for some  $\delta > \frac{1}{2}\sqrt{\pi(A)/\pi(A^c)}$  and thus  $\|P_x^{s+s'} - \pi\|_{2,\pi} \geq \delta\sqrt{\pi(A^c)/\pi(A)} > 1/2$ ).  $\square$

*Proof of Proposition 7.2:* Fix some leaf  $x$ ,  $0 < \delta \leq 1/4$  and  $A \in \text{Con}_\delta$ . Recall that  $\bar{A} = A^c \setminus D_\delta$ . Using the tree structure it is easy to see that for all  $s, s' \geq 0$

$$P_x[T_{A^c} > s+s'] \leq P_x[T_{\bar{A}} > s+s'] \leq P_x[T_{x_\delta} > s] + P_{x_\delta}[T_{\bar{A}} > s'] \leq P_x[T_{x_\delta} > s] + P_{\pi_{\mathcal{T}_{x_\delta}}}[T_{\bar{A}} > s']$$

and so by (3.1), the def. of  $b_x$  and the fact that  $\pi_{V \setminus \bar{A}}(\mathcal{T}_{x_\delta}) > 1/2$  (as  $\pi(V \setminus \bar{A}) < 2\delta < 2\pi(\mathcal{T}_{x_\delta})$ )

$$\begin{aligned} P_x[T_{A^c} > b_x + 3\kappa + 10t_{\text{rel}}] &\leq P_x[T_{x_\delta} > b_x] + P_{\pi_{\mathcal{T}_{x_\delta}}}[T_{\bar{A}} > 3\kappa + 10t_{\text{rel}}] \\ &< P_x[T_{x_\delta} > b_x] + 2P_{\pi_{V \setminus \bar{A}}}[T_{\bar{A}} > 3\kappa + 10t_{\text{rel}}] \leq \delta^3/4 + \delta^3/4 = \delta^3/2. \end{aligned} \quad \square$$

*Proof of Proposition 7.3:* By [2, Corollary 5.5] we have that  $\mathbb{E}_x[T_{x_\delta}] \leq \tau_1 + C_5\sqrt{\tau_1 t_{\text{rel}}}$  and hence it suffices to show that

$$\forall t \in [0, 2\mathbb{E}_x[T_{x_\delta}]], \quad P_x[T_{x_\delta} \geq \mathbb{E}_x[T_{x_\delta}] + t] \leq \exp[-t^2\lambda(D_\delta)/(8\mathbb{E}_x[T_{x_\delta}])]. \quad (7.6)$$

$$\forall t \geq 2\mathbb{E}_x[T_{x_\delta}], \quad P_x[T_{x_\delta} \geq \mathbb{E}_x[T_{x_\delta}] + t] \leq \exp[-\lambda(D_\delta)t/4]. \quad (7.7)$$

Indeed, if  $t_1 := 8\sqrt{\mathbb{E}_x[T_{x_\delta}]/\alpha_x(\delta)} \leq 2\mathbb{E}_x[T_{x_\delta}]$  then by (7.6)  $P_x[T_{x_\delta} \geq \mathbb{E}_x[T_{x_\delta}] + t_1] \leq \delta^3/4$ . Otherwise,  $t_2 := 32/\alpha_x(\delta) > 2\mathbb{E}_x[T_{x_\delta}]$ , and by (7.7),  $P_x[T_{x_\delta} \geq \mathbb{E}_x[T_{x_\delta}] + t_2] \leq \delta^3/4$ .

We note that (7.6) is essentially Lemma 5.8 in [2]. We start with an auxiliary calculation

**Claim 7.5.** Fix some leaf  $x$  and  $\delta \in (0, 1/4]$ . Let  $D_\delta$  be the connected component of  $x$  in  $\mathcal{T} \setminus \{x_\delta\}$ . Let  $y \in D_\delta$  and  $z$  be its parent. Then for all  $\beta \leq \lambda(D_\delta)/2$  we have that

$$\mathbb{E}_y[e^{\beta T_z}] \leq 1 + \mathbb{E}_y[T_z]\beta(1 + 2\beta/\lambda(D_\delta)) \leq e^{\mathbb{E}_y[T_z]\beta(1+2\beta/\lambda(D_\delta))}. \quad (7.8)$$

*Proof of (7.8):* Let  $\Phi(\mathcal{T}_y) := \frac{\pi(y)P(y,z)}{\pi(\mathcal{T}_y)}$ . Let  $f$  and  $g$  be the density functions of  $T_z$  started from  $y$  and  $\pi_{\mathcal{T}_y}$ , resp.. By Kac formula (c.f. [2, Proposition 5.6] or [1, (2.23)]),

$$\forall t \geq 0, \quad g(t) = \Phi(\mathcal{T}_y)P_y[T_z > t], \quad \text{and hence} \quad \Phi(\mathcal{T}_y)\mathbb{E}_y[T_z] = 1. \quad (7.9)$$

Recall that by (3.1) the law of  $T_z$  starting from  $\pi_{\mathcal{T}_y}$  is stochastically dominated by the Exponential distribution with parameter  $\lambda(\mathcal{T}_y) \geq \lambda(D_\delta)$  and so for every non-decreasing function  $k$  we have that  $\int_0^\infty k(t)g(t)dt \leq \int_0^\infty k(t)\lambda(D_\delta)e^{-\lambda(D_\delta)t}dt$ . Finally by (7.9)

$$\begin{aligned} \mathbb{E}_y[e^{\beta T_z}] - 1 &= \int (e^{\beta t} - 1)f(t)dt = \int \beta e^{\beta t}P_y[T_z > t]dt = \mathbb{E}_y[T_z] \int \beta e^{\beta t}g(t)dt \\ &= \beta \mathbb{E}_y[T_z] \int e^{\beta t}\lambda(D_\delta)e^{-\lambda(D_\delta)t}dt = \frac{\beta \mathbb{E}_y[T_z]\lambda(D_\delta)}{\lambda(D_\delta) - \beta} \leq \mathbb{E}_y[T_z]\beta(1 + 2\beta/\lambda(D_\delta)), \end{aligned}$$

where we used  $\beta \leq \lambda(D_\delta)/2$  to deduce that  $\frac{\lambda(D_\delta)}{\lambda(D_\delta) - \beta} = 1 + \frac{\beta}{\lambda(D_\delta) - \beta} \leq 1 + \frac{2\beta}{\lambda(D_\delta)}$ .  $\square$

We now return to conclude the proofs of (7.6)-(7.7). Let  $t \in [0, 2\mathbb{E}_x[T_{x_\delta}]]$ . Set  $\beta = \frac{t\lambda(D_\delta)}{4\mathbb{E}_x[T_{x_\delta}]}$  (note that  $\beta \leq \lambda(D_\delta)/2$ ). Let the path from  $x$  to  $x_\delta$  be  $(y_1 = x, \dots, y_r = x_\delta)$ . Observe that starting from  $x$  we have that  $T_{x_\delta} = \sum_{i=2}^r T_{y_i} - T_{y_{i-1}}$ . By the Markov property the terms in the sum are independent and  $T_{y_i} - T_{y_{i-1}}$  is distributed as  $T_{y_i}$  started from  $y_{i-1}$ . Denote  $\mu_i := \mathbb{E}_{y_{i-1}}[T_{y_i}]$  and  $\mu := \sum_{i=2}^r \mu_i = \mathbb{E}_x[T_{x_\delta}]$ . By (7.8), independence and our choice of  $\beta$

$$P_x[T_{x_\delta} \geq \mu + t] \leq e^{-\beta(\mu+t)} \prod_{i=2}^r \mathbb{E}_{y_{i-1}}[e^{\beta T_{y_i}}] \leq e^{-\beta(\mu+t)} \prod_{i=2}^r e^{\mu_i \beta(1+2\beta/\lambda(D_\delta))} = e^{-t^2 \lambda(D_\delta)/(8\mu)}.$$

The proof of (7.7) is analogous, now with the choice  $\beta = \lambda(D_\delta)/2$ .  $\square$

## 8 Open Problems

The *modified Log-Sobolev constant* is defined as

$$c_{\text{MLS}} := \inf_{f \in \mathbb{R}^\Omega} \mathcal{E}(e^f, f) / \text{Ent}_\pi(e^f).$$

The following question suggests a natural extension of Theorem 1.2. Recall that under reversibility  $1/c_{\text{LS}} \leq 2\tau_\infty$  and  $\lambda^{-1} \log 2 \leq \tau_1$  (e.g. [12, Lemma 20.11]). The following question asks whether a similar relation holds between  $c_{\text{MLS}}$  and  $\tau_{\text{Ent}}$ .

**Question 8.1.** *Is it the case that  $1/c_{\text{MLS}} \leq C\tau_{\text{Ent}}$  for some absolute constant  $C$ ?*

**Question 8.2.** *Recall that under reversibility  $\tau_2 \leq \rho + C/c_{\text{LS}}$ . Is it true that under reversibility  $\tau_{\text{Ent}} \leq \rho_{\text{Ent}} + C/c_{\text{MLS}}$ ?*

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